Exercise 9

The half-life of cesium-137 is 30 years. Suppose we have a 100-mg sample.

(a) Find the mass that remains after t years.

(b) How much of the sample remains after 100 years?

(c) After how long will only 1 mg remain?

Solution

Part (a)

Assume that the rate of mass decay is proportional to the amount of mass remaining at any given time.

$$\frac{dm}{dt} \propto -m$$

There's a minus sign here because mass is being lost as time increases. Change the proportionality to an equation by introducing a (positive) constant k.

$$\frac{dm}{dt} = -km$$

Divide both sides by m.

$$\frac{1}{m}\frac{dm}{dt} = -k$$

Rewrite the left side by using the chain rule.

$$\frac{d}{dt}\ln m = -k$$

The function you have to differentiate to get $-k$ is $-kt + C$, where C is any constant.

$$\ln m = -kt + C$$

Exponentiate both sides.

$$e^{\ln m} = e^{-kt+C}$$

$$m(t) = e^C e^{-kt}$$

Use a new constant m_0 for e^C.

$$m(t) = m_0 e^{-kt} \tag{1}$$

Use the fact that cesium-137 has a half-life of 30 years to get k.

$$\frac{m_0}{2} = m_0 e^{-k(30)}$$

$$\frac{1}{2} = e^{-30k}$$

$$\ln\frac{1}{2} = \ln e^{-30k}$$

$$-\ln 2 = -30k \ln e$$

$$k = \frac{\ln 2}{30} \approx 0.0231049 \text{ year}^{-1}$$

Equation (1) then becomes

$$\begin{aligned} m(t) &= m_0 e^{-\left(\frac{\ln 2}{30}\right)t} \\ &= m_0 e^{\ln 2^{-t/30}} \\ &= m_0 (2)^{-t/30}. \end{aligned}$$

Use the fact that the mass is 100 milligrams initially to determine m_0.

$$m(0) = m_0 (2)^{-(0)/30} = 100 \quad \rightarrow \quad m_0 = 100$$

Therefore, the mass in milligrams after t years have passed is

$$m(t) = 100(2)^{-t/30}.$$

Part (b)

The mass remaining after 100 years is

$$m(100) = 100(2)^{-100/30} \approx 9.92126 \text{ mg}.$$

Part (c)

To find how long it takes the sample to decay to 1 mg, set $m(t) = 1$ and solve the equation for t.

$$m(t) = 1$$

$$100(2)^{-t/30} = 1$$

$$2^{-t/30} = 0.01$$

$$\ln 2^{-t/30} = \ln 0.01$$

$$\left(-\frac{t}{30}\right) \ln 2 = -\ln 100$$

$$t = \frac{30 \ln 100}{\ln 2} \approx 199.316 \text{ years}$$